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1976 J. Phys. A: Math. Gen. 9 659

(http://iopscience.iop.org/0305-4470/9/4/022)

J. Phys. A: Math. Gen., Vol. 9, No. 4, 1976. Printed in Great Britain. © 1976

On the Takahashi–Umezawa quantization of the external field problem for multi-mass fields

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Received 24 November 1975

Abstract. The Takahashi–Umezawa quantization method is applied to certain first-order field equations which exhibit a mass spectra and have diagonalizable coefficient matrices. The Bhabha equations are a special case. It is shown that in all cases the independent field components satisfy the free field commutation rules in the presence of any external field. However, such theories have other difficulties.

1. Introduction

In the Takahashi–Umezawa (Takahashi 1969) method for quantizing interacting fields, the Heisenberg field operator $\Psi(x)$ is given by

$$\Psi(x) = \psi(x/\sigma) - \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}^4x'[\epsilon(x_0 - x_0'), d(\partial)]\Delta(x - x')j(x') \tag{1.1}$$

where $j(x)$ is the source in the field equation

$$\Lambda(\partial)\Psi(x) = j(x) \tag{1.2}$$

and $d(\partial)$ is the usual Klein–Gordon divisor defined by

$$(a)\quad d(\partial)\Lambda(\partial) \equiv \Box - m^2 \qquad \text{for unique mass} \tag{1.3a}$$

$$(b)\quad d(\partial)\Lambda(\partial) \equiv \prod_{i=1}^{N} (\Box - m_i^2) \qquad \text{for multi-mass.} \tag{1.3b}$$

The quantity $\epsilon(x_0 - x_0')$ is the sign function

$$\epsilon(x_0 - x_0') = \begin{cases} 1 & x_0 > x_0' \\ -1 & x_0 < x_0' \end{cases}$$

and $\Delta(x - x')$ is a generalization of the Schwinger solution of the Klein–Gordon equation which, in the general case of multi-mass fields, is given by (Baisya 1970)

$$\Delta(x - x') = \sum_{i=1}^{N} \frac{\Delta_{m_i}(x - x')}{\prod_{j\neq i}(m_i^2 - m_j^2)} \tag{1.4}$$

where

$$(\Box - m_i^2)\Delta_{m_i}(x - x') = \delta^{(4)}(x - x') \tag{1.5}$$

and then

$$\prod_{i=1}^{N} (\Box - m_i^2)\Delta(x-x') = \delta^{(4)}(x-x'). \tag{1.6}$$

The notation x/σ means that the spacelike surface σ passes through the point x. The interaction picture field $\psi(x/\sigma)$ then satisfies the free field equations and (anti-) commutation rules:

$$[\psi(x/\sigma), \hat{\psi}(x'/\sigma)]_{\pm} = \mathrm{i}\, d(\partial)\Delta(x-x'). \tag{1.7}$$

Here the adjoint field $\hat{\psi}(x)$ is defined by the $\hat{\psi}(x) = \epsilon(i)\psi_i^{+}(x)\eta$ where $\epsilon(i) = +1$ or -1 in order to make the free field energy a positive definite quantity and the $\psi_i(x)$ are the field operators corresponding to mass states m_i. η is the usual Hermitian operator satisfying

$$[\eta\Lambda(\partial)]^{+} = \eta\Lambda(-\overleftarrow{\partial})$$

where + denotes the Hermitian adjoint.

Clearly, theories for which

$$\Psi(x) = \psi(x/\sigma) \tag{1.8}$$

are the least offensive from the point of view of the Takahashi–Umezawa method, although from the many examples of such theories which now exist (Nagpal 1974), they invariably suffer from some problem such as indefiniteness of energy, or charge, or non-Hermiticity of the interaction Hamiltonian. If (1.8) holds then the Heisenberg field $\Psi(x)$ satisfies the free field (anti-) commutation rules.

It was originally thought that (1.8) held only for the simplest scalar and spinor fields (Takahashi and Umezawa 1964), but Baisya (1970, 1971) showed that it extended to various spin-$\frac{3}{2}$ theories with multiple masses and later Nagpal extended it to the Bhabha multi-mass equations (Nagpal 1973, 1974). These equations of Bhabha (Bhabha 1945) are based on the first-order matrix differential equation

$$\Lambda(\partial)\psi = (\beta_\mu\partial^\mu + m)\psi(x) = 0 \tag{1.9}$$

in which $\psi(x)$ carries a reducible representation of $\mathscr{L}_{\mathrm{p}}$ (proper Lorentz group) given by the reduction of an irreducible representation of the Lorentz group in five dimensions. The basic feature of these equations is that the matrix β_0 satisfies the minimal equation

$$(\beta_0^2+\tfrac{1}{4})(\beta_0^2+\tfrac{9}{4})\ldots(\beta_0^2+S^2) = 0 \tag{1.10}$$

for half-odd integer spin S, and

$$\beta_0(\beta_0^2+1)(\beta_0^2+4)\ldots(\beta_0^2+S^2) = 0 \tag{1.11}$$

for integer spin S. Nagpal shows that in both cases the dynamically independent fields satisfy (1.8).

Nagpal also discusses the propagation of the Bhabha fields in an external electromagnetic field and shows that they propagate causally. Amar and Dozzio (1975) considered the propagation of general theories of type (1.9) in an electromagnetic field and showed that a sufficient condition for causal propagation is that β_0 satisfies the minimal equation

$$\beta_0^r \prod_{i=1}^{N} (\beta_0^2+\lambda_i^2)^{r_i} = 0 \qquad r = 0 \text{ or } 1. \tag{1.12}$$

The distinguishing feature of such theories is that the sub-block of β_0 corresponding to the zero eigenvalues is diagonalizable. The case $r>1$ characterizes what Amar and Dozzio call type (c) constraints (Nagpal calls them secondary constraints) in the theory. It is only these types of constraint which can lead to causality problems. Capri and Shamaly (1973) have given an example of a causal spin-1 theory which contains type (c) constraints and in fact satisfies a minimal equation $\beta_0^3(\beta_0^2+1)=0$. So theories with type (c) constraints cannot be ruled out on the grounds of causality.

It is well known that theories characterized by (1.12) have vanishing charge or energy for those mass states $\pm m/\lambda_i$ for which $r_i>1$, and so the most general form for the minimal equation of β_0 for non-trivial theories is

$$\beta_0^r \prod_{i=1}^{N} (\beta_0^2+\lambda_i^2)=0 \tag{1.13}$$

i.e. the sub-blocks of β_0 corresponding to non-zero eigenvalues are diagonalizable (Speer 1969). For some theories r is related to the maximum spin contained in the theory, but in general it is a matter of choice (Glass 1971). Effectively, Amar and Dozzio's results show that all theories for which β_0 is diagonalizable are causal in the presence of an external electromagnetic field. The object of this note is to show that (1.8) also extends to such theories—i.e. it holds for all theories for which β_0 satisfies

$$\beta_0^r \prod_{i=1}^{N} (\beta_0^2+\lambda_i^2)=0 \qquad r=0 \text{ or } 1 \tag{1.14}$$

as minimal equation, with a slight modification in the case $r=1$ to single out the independent fields. The Bhabha equations, corresponding to (1.10), (1.11) are particular examples. Thus we generalize Nagpal's results and also derive them in a simpler way.

The essential idea we use in the case $r=0$ of (1.14) is that $d(\partial)$ does not contain sufficiently high-order derivatives to contribute to the right-hand side of (1.1). In the case $r=1$ of (1.14) the projection operator P_0 on to the null space of β_0 occurs as the term of highest-order derivative in $d(\partial)$, and it is only this term which can contribute to the integral on the right-hand side of (1.1).

2. The form of $d(\partial)$

The general form for the Klein–Gordon divisor was first given by Umezawa and Visconti (1956), for both cases (a) and (b) in (1.3). Explicity the expression is, for (1.3(b))

$$d(\partial)=\alpha_0+\alpha_\mu\partial^\mu+\alpha_{\mu_1\mu_2}\partial^{\mu_1}\partial^{\mu_2}+\ldots+\alpha_{\mu_1\mu_2\ldots\mu_L}\partial^{\mu_1}\partial^{\mu_2}\ldots\partial^{\mu_L} \tag{2.1}$$

where

$$\alpha_0=\frac{(-1)^N}{m}\prod_{i=1}^{N} m_i^2$$

$$\alpha_\mu=\frac{-\alpha_0\beta_\mu}{m}$$

$$\alpha_{\mu_1\mu_2}=\frac{(-1)^N}{m^3}\prod_{i=1}^{N} m_i^2\left(\beta_{\mu_1}\beta_{\mu_2}-\sum_{j=1}^{N}\frac{m^2}{m_j^2}g_{\mu_1\mu_2}\right)$$

$$\alpha_{\mu_1\mu_2\mu_3} = -\frac{\alpha_{\mu_1\mu_2}\beta_{\mu_3}}{m}$$

$$\alpha_{\mu_1\mu_2\mu_3\mu_4} = \frac{(-1)^N}{m^5}\prod_{i=1}^{N} m_i^2\left(\beta_{\mu_1}\beta_{\mu_2}\beta_{\mu_3}\beta_{\mu_4} - \sum_{j=1}^{N}\frac{m^2}{m_j^2}g_{\mu_1\mu_2}\beta_{\mu_3}\beta_{\mu_4} + \sum_{j\neq k}\frac{m^4}{2m_j^2m_k^2}g_{\mu_1\mu_2}g_{\mu_3\mu_4}\right)$$

$$\alpha_{\mu_1\ldots\mu_5} = -\frac{\alpha_{\mu_1\mu_2\mu_3\mu_4}\beta_{\mu_5}}{m}$$

etc, where $m_i = m/\lambda_i$, giving the relation between the non-zero eigenvalues of β_0 and the mass states.

We now consider separately the two minimal equations for β_0, given in (1.14).

2.1. $r=0$

β_0 satisfies

$$\prod_{i=1}^{N}(\beta_0^2+\lambda_i^2)=0 \tag{2.2}$$

as the minimal equation. It is easy to verify that in this case we can take the Klein–Gordon divisor to be

$$d(\partial) = \alpha_0 + \alpha_\mu\partial^\mu + \alpha_{\mu_1\mu_2}\partial^{\mu_1}\partial^{\mu_2} + \ldots + \alpha_{\mu_1\ldots\mu_{2N-1}}\partial^{\mu_1}\partial^{\mu_2}\ldots\partial^{\mu_{2N-1}} \tag{2.3}$$

with the coefficients $\alpha_{\mu_1\ldots\mu_2}$ as given in (2.1). Then (1.3(b)) becomes a consequence of the covariant form of (2.2):

$$\prod_{i=1}^{N}((\beta\cdot\partial)^2+\lambda_i^2\Box)\equiv 0.$$

Actually, all we need to know in this case is that the highest order of derivative in $d(\partial)$ is $2N-1$, where $2N$ is the number of (distinct) mass-charge states.

2.2. $r=1$

β_0 satisfies

$$\beta_0\prod_{i=1}^{N}(\beta_0^2+\lambda_i^2)=0 \tag{2.4}$$

as the minimal equation. This case is a little more complicated, because of the singular nature of β_0, but substituting (2.1) in (1.3(b)) we obtain for the highest-order derivative in $d(\partial)\Lambda(\partial)$:

$$\alpha_{\mu_1\ldots\mu_L}\beta_{\mu_{L+1}}\partial^{\mu_1}\ldots\partial^{\mu_L}\partial^{\mu_{L+1}}.$$

Now either $L+1=2N$ and (1.3(b)) gives

$$\alpha_{\mu_1\ldots\mu_L}\beta_{\mu_{L+1}}\partial^{\mu_1}\partial^{\mu_{L+1}}=(\Box)^{2N}$$

or $L+1=2N+1$ and

$$\alpha_{\mu_1\ldots\mu_{L+1}}\beta_{\mu_{L+1}}\partial^{\mu_1}\ldots\partial^{\mu_L}\partial^{\mu_{L+1}}\equiv 0 \tag{2.5}$$

(note that the identity here refers to the derivatives $\partial^{\mu_1}\dots\partial^{\mu_{L+1}}$). Since the first possibility would imply a minimal equation for β_μ of even degree, the case (2.4) must correspond to the second possibility—the first in fact corresponds to the case (2.2). Thus (2.5) is the covariant form of (2.4), as is directly verified from (2.1). The important point however is that, on putting all indices μ_i in (2.5) equal to zero we get

$$\beta_0\alpha_{00\dots0}=0 \tag{2.6}$$

and comparing with (2.4) this means

$$\alpha_{00\dots0}=\kappa\prod_{i=1}^{N}(\beta_0^2+\lambda_i^2) \tag{2.7}$$

in other words, $\alpha_{0\dots0}$ is proprotional to the projection operator on to the null space of β_0. Thus, for theories satisfying (2.4), $d(\partial)$ can be written in the form

$$d(\partial)=\alpha_0+\alpha_\mu\partial^\mu+\dots+\alpha_{\mu_1\dots\mu_{2N-1}}\partial^{\mu_1}\dots\partial^{\mu_{2N-1}}+\frac{1}{m}P(\beta,\partial) \tag{2.8}$$

where $P(\beta,\partial)$ is the covariant form of the projection operator:

$$P_0=\frac{1}{\Pi_i\lambda_i^2}\prod_{i=1}^{N}(\beta_0^2+\lambda_i^2) \tag{2.9}$$

so

$$\begin{aligned}P(\beta,\partial)&=\frac{1}{\Pi_i\lambda_i^2}\prod_{i=1}^{N}((\beta,\partial)^2+\lambda_i^2\Box)\\&=\frac{(-1)^N}{\Pi_i\lambda_i^2}(\beta_{\mu_1}\beta_{\mu_2}-\lambda_1^2g_{\mu_1\mu_2})(\beta_{\mu_3}\beta_{\mu_4}-\lambda_2^2g_{\mu_3\mu_4})\dots\\&\quad\times(\beta_{\mu_{2N-1}}\beta_{\mu_{2N}}-\lambda_n^2g_{\mu_{2N-1}\mu_{2N}})\partial^{\mu_1}\dots\partial^{\mu_{2N}}.\end{aligned} \tag{2.10}$$

The important point about (2.8) is the occurrence of the projection operator $P(\beta,\partial)$ as the term of highest-order derivative. Strictly, $P(\beta,\partial)$ is only a projection operator on the mass shell, but this does not affect the arguments used.

3. A preliminary result

To evaluate the integral on the right-hand side of (1.1) we use the results of appendix D in Takahashi (1969) which provides formulae for $\frac{1}{2}[\epsilon(x_0-x_0'),F(\partial)]\Delta(x-x')$. Specifically, we use the result of Katayama (1953)

$$\begin{aligned}&\tfrac{1}{2}[\epsilon(x_0-x_0'),\partial^{\mu_1}\dots\partial^{\mu_l}]\Delta(x-x')\\&=-\frac{1}{K_+K_-}\Big(\partial^{\mu_1}\dots\partial^{\mu_l}-\frac{1}{K_++K_-}\{K_+(\partial^{\mu_1}-n^{\mu_1}K_-)\dots(\partial^{\mu_l}-n^{\mu_l}K_-)\\&\quad+K_-(\partial^{\mu_1}+n^{\mu_1}K_+)\dots(\partial^{\mu_l}+n^{\mu_l}K_+)\}\Big)\delta^{(4)}(x-x')\end{aligned} \tag{3.1}$$

where

$$K_+=(\nabla^2-m^2)^{1/2}+n\,.\,\partial \qquad K_-=(\nabla^2-m^2)^{1/2}-n\,.\,\partial$$

and $(\Box-m^2)\Delta(x-x')=0$. n_μ is a timelike unit vector. However, it is more convenient to have the right-hand side as a power series in the mass m. We find, with obvious notation:

$$\begin{aligned}
&\tfrac{1}{2}[\epsilon(x_0-x_0'),\partial^{\mu_1}\ldots\partial^{\mu_l}]\Delta(x-x')\\
&\quad=\sum_{k=1}^{l}\frac{K_-K_+^k+(-1)^kK_+K_-^k}{K_+K_-(K_++K_-)}\sum(n)^k(\partial)^{l-1}\delta^{(4)}(x-x')\\
&\quad=\sum_{k=1}^{l}\frac{K_+^{k-1}-(-1)^{k-1}K_-^{k-1}}{K_++K_-}\sum(n)^k(\partial)^{l-k}\delta^{(4)}(x-x')\\
&\quad=\sum_{j=0}^{l-1}\frac{K_+^j-(-1)^jK_-^j}{K_++K_-}\sum(n)^{j+1}(\partial)^{l-j-1}\delta^{(4)}(x-x')\\
&\quad=\sum_{j=0}^{l-1}\frac{(a+b)^j-(-1)^j(a-b)^j}{2a}\sum(n)^{j+1}(\partial)^{l-j-1}\delta^{(4)}(x-x')
\end{aligned}$$

where $a=(\nabla^2-m^2)^{1/2}$, $b=n\,.\,\partial$

$$=\sum_{j=0}^{l-1}\sum_{q=0}^{j}\binom{j}{q}a^{j-q-1}b^q\frac{[1-(-1)^{j+q}]}{2}\sum(n)^{j+1}(\partial)^{l-j-1}\delta^{(4)}(x-x'). \tag{3.2}$$

The terms in this sum for which $j+q$ is even vanish and only those terms for which j and q have different parity remain. Then $j-q-1$ is even and only even powers of $a=(\nabla^2-m^2)^{1/2}$ occur. The highest power present is a^{l-2} if l is even and a^{l-3} if l is odd. So the right-hand side of (3.1) is a polynomial in m^2 of degree $(l-2)/2$ if l is even and $(l-3)/2$ if l is odd. We therefore write

$$\tfrac{1}{2}[\epsilon(x_0-x_0'),\partial^{\mu_1}\ldots\partial^{\mu_l}]\Delta(x-x')
=\begin{cases}\displaystyle\sum_{j=0}^{(l-2)/2} b_j(n,\partial)(m^2)^j\delta^{(4)}(x-x') & \text{if } l \text{ is even}\\[2ex]
\displaystyle\sum_{j=0}^{(l-3)/2} c_j(n,\partial)(m^2)^j\delta^{(4)}(x-x') & \text{if } l \text{ is odd,}\end{cases} \tag{3.3}$$

the b_j and c_j being independent of m^2. We will only need to know $b_{(l-2)/2}$ explicitly and this is easily read off from the $j=l-1$, $r=0$ term of (3.2) as:

$$b_{(l-2)/2}=(-1)^{(l-2)/2}n^{\mu_1}\ldots n^{\mu_l}. \tag{3.4}$$

4. Quantization of the interacting field

Again we treat the cases (2.2) and (2.4) separately.

4.1. $r=0$

(2.3) gives the expression for $d(\partial)$ in this case and substituting this we find

$$
\begin{aligned}
&\tfrac{1}{2}[\epsilon(x_0-x_0'),\, d(\partial)]\Delta(x-x')\\
&\quad=\sum_{l=1}^{2N-1}\alpha_{\mu_1\ldots\mu_l}\tfrac{1}{2}[\epsilon(x_0-x_0'),\,\partial^{\mu_1}\ldots\partial^{\mu_l}]\Delta(x-x')\\
&\quad=\sum_{l=1}^{2N-1}\alpha_{\mu_1\ldots\mu_l}\sum_{i=1}^{N}\frac{\frac{1}{2}[\epsilon(x_0-x_0'),\,\partial^{\mu_1}\ldots\partial^{\mu_l}]\Delta_{m_i}(x-x')}{\Pi_{i\neq j}(m_i^2-m_j^2)}
\end{aligned}
$$

from (1.4). We now use (3.3) to rewrite this in the form

$$\sum_{k=0}^{N-2}A_k(\beta,n,\partial)\sum_{i=1}^{N}\frac{(m_i^2)^k}{\Pi_{i\neq j}(m_i^2-m_j^2)} \tag{4.1}$$

where the A_k are independent of the i summation, although they do contain the masses m_i in a symmetric fashion (these occur in the $\alpha_{\mu_1\ldots\mu_l}$, which do not enter the i summation). Finally, we use the algebraic identities (Nagpal 1974):

$$
\begin{aligned}
&\sum_{i=1}^{N}\frac{m_i^\alpha}{\Pi_{i\neq j}(m_i^2-m_j^2)}=0 \qquad \alpha=0,2,\ldots(2N-4)\\
&\sum_{i=1}^{N}\frac{m_i^{2N-2}}{\Pi_{i\neq j}(m_i^2-m_j^2)}=1 \qquad N\geqslant 2
\end{aligned} \tag{4.2}
$$

to note that (4.1) vanishes. So for these theories, satisfying (2.2), we obtain (1.8). The interaction commutation rules are the same as the free field ones. This is quite independent of the source $j(x)$.

4.2. $r=1$

In this case $d(\partial)$ is given by (2.8):

$$d(\partial)=\alpha_0+\sum_{l=1}^{2N-1}\alpha_{\mu_1\ldots\mu_l}\partial^{\mu_1}\ldots\partial^{\mu_l}+\frac{1}{m}P(\beta,\partial).$$

Substituting in $\frac{1}{2}[\epsilon(x_0-x_0'),\, d(\partial)]\Delta(x-x')$, the first two terms vanish exactly as in (4.1) and we obtain

$$
\begin{aligned}
&\tfrac{1}{2}[\epsilon(x_0-x_0'),\, d(\partial)]\Delta(x-x')\\
&\quad=\frac{1}{m}\tfrac{1}{2}[\epsilon(x_0-x_0'),\, P(\beta,\partial)]\Delta(x-x')\\
&\quad=\frac{1}{m\Pi_i\lambda_i^2}(\beta_{\mu_1}\beta_{\mu_2}-\lambda_1^2g_{\mu_1\mu_2})\ldots(\beta_{\mu_{2N-1}}\beta_{\mu_{2N}}-\lambda_N^2g_{\mu_{2N-1}\mu_{2N}})\\
&\qquad\times\tfrac{1}{2}[\epsilon(x_0-x_0'),\,\partial^{\mu_1}\ldots\partial^{\mu_{2N}}]\Delta(x-x')\\
&\quad=\frac{1}{m}P(\beta,\sqrt{g})\times\sum_{i=1}^{N}\frac{\frac{1}{2}[\epsilon(x_0-x_0'),\,\partial^{\mu_1}\ldots\partial^{\mu_{2N}}]\Delta_{m_i}(x-x')}{\Pi_{i\neq j}(m_i^2-m_j^2)}
\end{aligned}
$$

with obvious notation

$$
\begin{aligned}
&\quad=\frac{1}{m}P(\beta,\sqrt{g})\sum_{j=0}^{N-1}b_j(n,\partial)\sum_{i=1}^{N}\frac{m^{2j}}{\Pi_{i\neq j}(m_i^2-m_j^2)}\delta^{(4)}(x-x')\\
&\quad=\frac{1}{m}P(\beta,\sqrt{g})b_{N-1}(n,\partial)\cdot 1\cdot\delta^{(4)}(x-x')
\end{aligned}
$$

using the identities (4.2)

$$=\frac{1}{m\Pi_i\lambda_i^2}(\beta_{\mu_1}\beta_{\mu_2}-\lambda_1^2 g_{\mu_1\mu_2})\ldots(\beta_{\mu_{2N-1}}\beta_{2N}-\lambda_n^2 g_{\mu_{2N-1}\mu_{2N}})$$
$$\times(-1)^{N-1}n^{\mu_1}\ldots n^{\mu_{2N}}\delta^{(4)}(x-x')$$

from (3.4)

$$=\frac{(-1)^{N-1}}{m}P(\beta, n)\delta^{(4)}(x-x').$$

Substituting in (1.1) we obtain:

$$\Psi(x)=\psi(x/\sigma)-\frac{(-1)^{N-1}}{m}\int_{-\infty}^{\infty} \mathrm{d}^4x' P(\beta, n)\delta^{(4)}(x-x')j(x')$$
$$=\psi(x/\sigma)+\frac{(-1)^N}{m}P(\beta, n)j(x)$$

which generalizes the result preceeding (2.16) of Nagpal (1974). We now use the operator $I-P(\beta, n)$ to project out the independent field components, and using

$$(I-P(\beta, n))P(\beta, n)=0$$

obtain

$$(I-P(\beta, n))\Psi(x)=(I-P(\beta, n))\psi(x/\sigma).$$

So the independent field components satisfy (1.8) for theories satisfying (2.4). Again, this result is independent of the source $j(x)$.

5. Conclusion

We have shown that for all theories in which β_0 is diagonalizable, (1.8) holds for the independent field components. This is independent of the source of the external field. Thus, as Amar and Dozzio found in the case of the causality problem, only type (c) constraints can lead to problems in the external field quantization due to normal dependent terms occurring in (1.1). However, just as in the causality problem, there is no obvious reason to suggest that type (c) constraints will always mean a violation of (1.8), or if it does that this still precludes a 'consistent' external field quantization. If type (c) constraints do exist, so that $r>1$ in (1.13), then it is no longer possible to define a good projection operator P_0 onto the null space of β_0 and $1-P_0$ cannot be used to isolate the independent components of the field. The Jordan block of zero eigenvalues of β_0 is not diagonal and the analysis of the constraints becomes very difficult, but this still does not mean that the independent field components will violate (1.8).

Further, the apparent simplicity of the theories considered here, free of type (c) constraints, is illusory, at least for higher spin. Such theories exhibit other problems. It is well known that for spin greater than one, theories with β_0 diagonalizable which are invariant under the complete Lorentz group ($\mathscr{L}_p$+reflections) and which are derivable from a real Lagrangian, cannot satisfy the correct definiteness properties on energy and charge without an indefinite metric (Gel'fand *et al* 1963). Also, it has been observed that the interaction Hamiltonian $H(x)=\hat{\psi}(x)j(x)$, obtained from the equation

$$[\psi(x/\sigma), H(x'/\sigma, n)] = \mathrm{i}\, d(\partial)\Delta(x-x')j(x')$$

using the free field commutation rules (1.7) and the fact (1.8) for the theories with β_0 diagonalizable, is in general non-Hermitian (Nagpal 1973). $H(x)$ can be made Hermitian using an indefinite metric but this can upset the definiteness of the energy or charge (Baisya 1970). Munczek (1967) has proposed a way round this problem by using a variant of the Lee–Yang ξ-limiting formalism for the massive vector-boson field, but the results are not conclusive in this respect. However, despite their difficulties, theories with β_0 diagonalizable have received a great deal of attention recently, because of the absence of type (c) constraints and the simplicity of their mass spectra (Iverson 1971). Also Hurley (1974) has proposed a formalism which may resolve the above problems and make it possible to incorporate such theories into a consistent scheme.

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